

r-modes in the Tolman VII solution

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The r-mode frequencies of the Tolman VII solution for the slowly rotating non-barotropic approximation within the low frequency regime are estimated. The relativistic correction to Newtonian r-mode calculations is shown as function of the tenuity $\frac{R}{M}$ and is shown to be significant only for very compact neutron stars.

I. INTRODUCTION

The possibility that the fluid motion of rotating neutron stars is unstable to the emission of gravitational radiation has motivated numerous recent studies of the so-called “r-mode” instability [1]. Although this is a relativistic effect, arising from perturbations of the spacetime, the majority of available work has been focused on Newtonian and post-Newtonian approximations. In the first-order (in the angular velocity Ω) relativistic treatment the eigenfrequencies are real and the modes are determined by one ordinary differential equation [2] in the low frequency regime for non-barotropic [3] stars. This singular [4] eigenvalue problem then gives rise to a continuous spectrum, but can also admit a discrete r-mode solution [5] (At higher frequencies the r-modes exhibit complex eigenfrequencies, losing their singular structure.) It has been further shown [6] that if the mode occurs inside the star, then it is associated with a diverging velocity perturbation. It has been shown that the r-modes of low-index realistic polytropic models exhibit this unphysical feature, while constant density models do not [6], suggesting that the r-mode instability might be a curiosity of unphysical models. It is more likely, however, that it is the mathematical description of the problem that is failing, not the underlying physics. Taking this view, we present a straightforward way to estimate the relativistic correction to Newtonian r-mode frequencies for realistic equations of state. Calculations are then performed in the slowly rotating counterpart to the Tolman VII solution - the exact solution of Einstein’s field equations that is currently the best fit to a neutron star [7]. We show that the maximum relativistic correction to Newtonian r-mode calculations based on this model is $< \sim 23\%$ for entirely causal solutions.

II. THE EIGENVALUE PROBLEM

The static spherically symmetric metric in “curvature” coordinates [8] is

$$ds^2 = ds_{\Gamma}^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \quad (1)$$

where ds_{Γ}^2 is a static Lorentzian 2-surface with coordinates (t, r) . The source of (1) is considered to be a perfect mathematical fluid (of isotropic pressure $p(r)$ and energy density $\rho(r)$) generated by radial streamlines of constant r . To first order in the angular velocity (Ω), a rotating star can be described by the stationary axisymmetric metric

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 - 2w(r) r^2 \sin^2(\theta) dt d\phi, \quad (2)$$

where the functions $\nu(r)$ and $\lambda(r)$ are identical to those in (1). The function $w(r)$ satisfies Hartle’s equation [9]

$$\frac{1}{r^4} \frac{d}{dr} \left(r^4 j \frac{d\bar{w}(r)}{dr} \right) + \frac{4}{r} \frac{dj}{dr} \bar{w}(r) = 0, \quad (3)$$

subject to the boundary condition

$$1 = \Omega = [\bar{w}(r) + \frac{R}{3} \frac{d\bar{w}(r)}{dr}]_{r=R}, \quad (4)$$

where $j = e^{-\frac{\nu(r)+\lambda(r)}{2}}$ and $\bar{w}(r) = \Omega - w(r)$. The axial modes for non-barotropic stars are determined by Kojima’s equation [2]

$$(\alpha - \bar{w}(r))(e^{-\lambda(r)} h_o'' - 4r\pi\Upsilon(r)h_o' - (8\pi\Upsilon(r) - 4\frac{M}{r^3} + \frac{l(l+1)}{r^2})h_o) + 16\alpha\pi\Upsilon(r)h_o + O(\Omega^2) + \dots = 0, \quad (5)$$

where $' \equiv \frac{d}{dr}$,

$$\Upsilon(r) = p(r) + \rho(r), \quad (6)$$

and h_o is a perturbation variable (l and m are spherical harmonic indices). Note that we have included higher-order corrections to this description. The problem is singular at r_s when $\bar{w}(r_s) = \alpha$, the eigenvalue, which is related to the mode frequency (σ) by

$$\sigma = -m\Omega(1 - \frac{2\alpha}{l(l+1)}). \quad (7)$$

Newtonian r-modes have $\alpha = 1$. The fluid velocity perturbation (u_s) vanishes at the boundary defined by the vanishing of the isotropic pressure. It is related to the perturbation variable by

$$u_s = \frac{2m\bar{w}(r)}{2m\bar{w}(r) - l(l+1)(\sigma + m\Omega)} e^{\frac{\lambda(r) - \nu(r)}{2}} h_o. \quad (8)$$

Kojima's equation (5) admits a continuous spectrum from the centre ($\bar{w}(0)$) to the boundary ($\bar{w}(R)$) [4], and may also admit discrete r-mode solutions [10]. The singular point corresponding to these solutions cannot be inside the star, as the velocity perturbation would diverge [6]. In this scenario, therefore, eigenvalue solutions are only physical if $\bar{w}(R) \leq \alpha \leq 1$, whereas many realistic equations of state yield $\alpha < \bar{w}(R)$. It is likely that this unphysical behaviour arises from the neglect of higher order terms near the r-mode, since the r-mode instability appears even in Newtonian theory [11]. These higher-order terms can only be significant when the $16\alpha\pi\Upsilon(r)h_o$ term in (5) is small. For realistic equations of state, $\rho(r)$ monotonically decreases through the solution, reaching a minimum at the boundary, which is defined by $p(R) = 0$. Hence, the eigenvalues must correspond to a value of $\bar{w}(r)$ close to the boundary of the solution. Therefore a useful approximation for computing the relativistic correction to Newtonian r-mode calculations for realistic equations of state is that $\alpha \approx \bar{w}(R)$.

III. THE TOLMAN VII SOLUTION

The Tolman VII solution [12] is one of only two [13] known physically reasonable exact spherically symmetric perfect fluid solutions of Einstein's equations where the energy density vanishes at the boundary [14]. The associated metric is

$$ds^2 = -\sin(\ln(\sqrt{1 - 5\frac{\xi^2}{\gamma} + 3\frac{\xi^4}{\gamma} + \frac{\xi 2\sqrt{3}}{\sqrt{\gamma}} - 5/6\frac{\sqrt{3}}{\sqrt{\gamma}})C^{-1}))^2 dt^2 + R^2(1 - 5\frac{\xi^2}{\gamma} + 3\frac{\xi^4}{\gamma})^{-1} dr^2 \quad (9)$$

where C is a number chosen to give zero energy density at the boundary [15] and is given by

$$C = \frac{2}{\sqrt{3\gamma}} \sqrt{\frac{12\gamma^{3/2} - 23\sqrt{\gamma} + 4\sqrt{3}\sqrt{\gamma(\gamma-2)}}{e^{4\arctan(\varsigma)}}}, \quad (10)$$

where

$$\varsigma = \frac{3\gamma^{3/2} - 6\sqrt{\gamma} + \tan(\ln(2))\sqrt{3}\sqrt{\gamma(\gamma-2)}}{-3\gamma^{3/2}\tan(\ln(2)) + 6\sqrt{\gamma}\tan(\ln(2)) + \sqrt{3}\sqrt{\gamma(\gamma-2)}}, \quad (11)$$

$\xi = \frac{r}{R}$, and γ is the tenuity (total radius to mass ratio, $\frac{R}{M}$). The solution is causal [16] for $\gamma > \sim 3.707$. The isotropic pressure is not a "elementary" function. In contrast, the energy density is simply

$$\rho(\xi) = \rho_c(1 - \xi^2), \quad (12)$$

where $\rho_c = \frac{15}{8\pi R^2\gamma}$. Despite its simplicity, this density profile (and the corresponding binding energy and moment of inertia determined from it) is in good agreement with realistic equations of state for neutron stars with $M > 1.2M_\odot$ [7], even in the star's intermediate regions. Another appealing feature of this solution is that it does not have a phase transition (in ρ) at the boundary. Since the Tolman VII solution can reproduce several features that are independent of the neutron star equation of state, we take the view that using it is the best way by which to gauge the relativistic correction to Newtonian r-mode calculations.

IV. RESULTS AND CONCLUSIONS

Numerical integrations of (3) in the Tolman VII solution give the plot shown in Figure 1. At the causal limit (subluminal adiabatic sound speed throughout the interior) the eigenvalue is $\alpha \approx \bar{w}(R) \approx 0.77$ which gives an estimate of 23% for the correction to the Newtonian r-mode calculation. The size of the correction then decreases monotonically with increasing γ . We find, therefore, that the difference between the r-mode frequencies of realistic slowing rotating non-barotropic neutron stars and those of Newtonian models in the low-frequency approximation, is only significant for very compact neutron stars.

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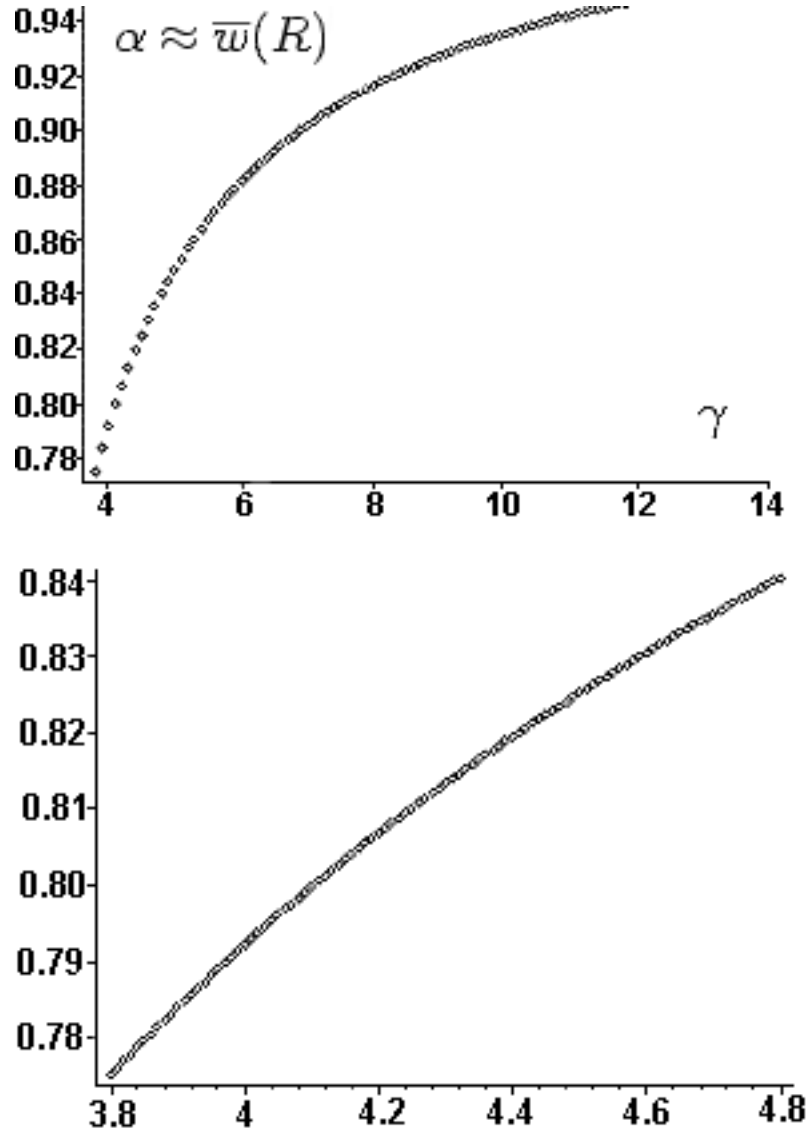


FIG. 1. Numerical integration of (3) in the Tolman VII solution. The solution is entirely causal for $\gamma > \sim 3.707$. An enlarged view of the compact region is shown. At the causal limit (subluminal adiabatic sound speed throughout the interior) the eigenvalue is $\alpha \approx \bar{w}(R) \approx 0.77$ which gives an estimate of 23% for the maximum correction to the Newtonian r-mode calculation. Note that the size of the correction then decreases monotonically with increasing γ .